All–order uniform momentum bounds for the massless ϕ^4 theory in four dimensional Euclidean space

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Abstract: A panoramic overview is given, of a theorem [1] establishing physical and uniform bounds on the Fourier–transformed Schwinger functions of a massless ϕ^4 theory in four Euclidean dimensions, at any loop order in perturbation theory. (Talk given by RG at the Oberwolfach workshop "The Renormalization Group", March 13th - March 19th, 2011.)

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All–order uniform momentum bounds for the massless ϕ^4 theory in four dimensional Euclidean space

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(joint work with Christoph Kopper,[1])

A panoramic overview is given, of a theorem [1] establishing physical and uniform bounds on the Fourier–transformed Schwinger functions of a massless ϕ^4 theory in four Euclidean dimensions, at any loop order in perturbation theory.

The first step to set up the perturbative framework is to specify a free quantum theory describing a massless scalar field by fixing a centered Gaussian measure on $\mathcal{S}'(\mathbb{R}^4)$, $\mu_{\hbar C_R^{\Lambda,\Lambda_0}}$, whose covariance $\hbar C_R^{\Lambda,\Lambda_0}(x,y) := \hbar\chi_R(x)\,\chi_R(y)\,C^{\Lambda,\Lambda_0}(x-y)$ is assumed to be a distribution in $\mathcal{S}'(\mathbb{R}^8)$ acting as a positive bilinear form on test functions. $\hbar > 0$ denotes the variable of the formal perturbative series. The short–distance behavior (smoothness) of $C^{\Lambda,\Lambda_0}(x)$ as a function is controlled by $\Lambda_0 > 0$ (known as ultra–violet, UV, cutoff), while the long–distance regularity is controlled by $0 < \Lambda \le \Lambda_0$ (infra–red, IR, cutoff). C^{Λ_0,Λ_0} vanishes. When Λ_0 tends to infinity and Λ tends to zero, $C^{\Lambda,\Lambda_0}(x)$ approaches the standard free propagator $\left\langle x \left| \partial^{-2} \right| 0 \right\rangle$. For any $R > 0$, the non–negative function $\chi_R \in \mathcal{C}_c^\infty\left(\mathbb{R}^4\right)$ satisfies the "finite–volume" constraint $\chi_R(x) = 1$ for any $|x| \le R$.

For any $\mathtt{N} \in \mathbb{N}$, and any $\mathtt{L} \in \mathbb{N}_0$ the Schwinger functions in momentum space are defined by

$$
(1)\quad \hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{\mathtt{N},\mathtt{L}}(p_{[\mathtt{N}-1]}) := \lim_{R\to\infty}\left(\left(\frac{1}{\mathtt{L}!}\frac{\partial^{\mathtt{L}}}{\partial\hbar^{\mathtt{L}}}\right)_{\hbar=0}\left(\frac{\delta}{\delta\varphi(0)}\prod_{e=1}^{\mathtt{N}-1}\int \mathrm{d}^4x_e\; e^{-\mathrm{i}x_e p_e}\frac{\delta}{\delta\varphi(x_e)}\right)_{\varphi=0}\left(-\hbar\log\left(\int \mathrm{d}\mu_{\hbar C_R^{\Lambda,\Lambda_0}}(\phi)e^{-\frac{1}{\hbar}S^{\mathrm{int}}(\phi+\varphi)}\Big/\int \mathrm{d}\mu_{\hbar C_R^{\Lambda,\Lambda_0}}(\phi)e^{-\frac{1}{\hbar}S^{\mathrm{int}}(\phi)}\right)\right)\right),
$$

where: $[a] := [1:b]$, $[a:b] := \{n\in\mathbb{Z}|a\le n\le b\}$, and $p_{[n]} := (p_1,\cdots,p_n)$. In (1), the interaction action $S^{\mathrm{int}}(\varphi)$ is defined by

$$
(2)\qquad S^{\mathrm{int}}(\varphi) := \int \mathrm{d}^4x\left(A(\hbar)\,\frac{(\partial\varphi(x))^2}{2} + B_2(\hbar)\frac{\varphi(x)^2}{2} + B_4(\hbar)\frac{\varphi(x)^4}{4!}\right),
$$

where A, B_2, B_4 are formal series in $\hbar$, whose coefficients are fixed order by order by appropriate renormalization conditions, in such a way that the "UV+IR limit" $\lim_{\Lambda_0\to\infty}\lim_{\Lambda\to 0^+}\hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{\mathtt{N},\mathtt{L}}$ exists in $\mathcal{S}'(\mathbb{R}^{4(\mathtt{N}-1)})$ for all $\mathtt{N},\mathtt{L}$. In particular, it turns out for a massless theory that A, B_2 are of order $O(\hbar)$, while $B_4 = g_0 + O(\hbar)$. From (1) and (2) it follows that $\hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{2,0}$ and all $\hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{\mathtt{N},\mathtt{L}}$ with odd $\mathtt{N}$ vanish.

The UV+IR limit of $\hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{\mathtt{N},\mathtt{L}}$ is a regular function only at non–exceptional momenta, see e.g. [2]. (A collection of four vectors $p_{[\mathtt{N}-1]}$ is said *exceptional* iff it exists a non–empty $\mathbb{S}\subseteq[\mathtt{N}-1]$ such that $\sum_{e\in\mathbb{S}} p_e = 0$.)

Any Schwinger function $\hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{\mathtt{N},\mathtt{L}}$ defined in (1) can be computed from the standard weighted sum of all Feynman amplitudes proportional to $\hbar^{\mathtt{L}}$, obtained via Feynman rules from an appropriate set of connected amputated graphs with $\mathtt{N}$ external lines. Each such set includes all graphs with vertices of coordination number 4 and loop

number $\mathtt{L}$. The word "amputated" means that Feynman rules do not associate any factor to the external lines.

Schwinger functions satisfy the "Polchinski" renormalization group (RG) flow equations, [3] (see [4] for an introduction), which in their perturbative form read:

$$(3)\quad \partial_\Lambda \hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{\mathtt{N},\mathtt{L}}\left(p_{[\mathtt{N}-1]}\right) = \mathcal{F}^{\Lambda,\Lambda_0}_{\mathtt{N},\mathtt{L},w} := \Bigg(\frac{1}{2}\int \frac{\mathrm{d}^4\ell}{(2\pi)^4}\,\partial_\Lambda \hat{C}^{\Lambda,\Lambda_0}(\ell)\ \hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{\mathtt{N}+2,\mathtt{L}-1}\left(p_{[\mathtt{N}-1]},-\ell,\ell\right) \\ -\frac{1}{2}\sum_{\substack{\mathcal{E}'\uplus\mathcal{E}''=[0:\mathtt{N}-1]\\ \mathtt{L}'+\mathtt{L}''=\mathtt{L}}} \partial_\Lambda \hat{C}^{\Lambda,\Lambda_0}\left(\textstyle\sum_{e\in\mathcal{E}'} p_e\right)\ \hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{\mathtt{N}',\mathtt{L}'}\left(p_{\mathcal{E}'}\right)\ \hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{\mathtt{N}'',\mathtt{L}''}\left(p_{\mathcal{E}''}\right)\Bigg),$$

where $\mathtt{N}' := |\mathcal{E}'|+1$, $\mathtt{N}'' := |\mathcal{E}''|+1$, $p_0 := -\sum_{e\in[\mathtt{N}-1]} p_e$, and the sum on the r.h.s. of (3) runs over all disjoint (possibly empty) sets $\mathcal{E}', \mathcal{E}''$ whose union gives $[0:\mathtt{N}-1]$, as well as over all non–negative integers $\mathtt{L}', \mathtt{L}''$ whose sum gives $\mathtt{L}$.

When the field has a mass $m>0$, it is not difficult to use the RG equations to bound Schwinger functions in momentum space (see e.g. [4]). Such bounds are simple but clearly unphysical because they depend polynomially on external momenta; moreover, they diverge when the mass vanishes and the IR limit is taken. More physical bounds have been proved in the massive case, [5].

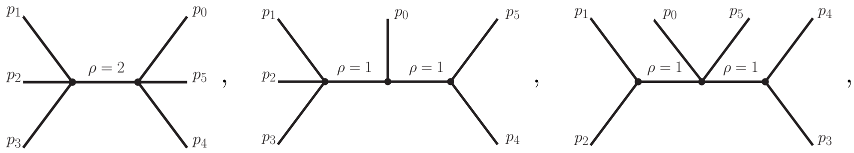
The goal of the "existence and boundedness theorem" in [1] is to extend the ideas in [5] to obtain physical, uniform bounds for the massless case. The theorem assumes that the Fourier–transformed covariance $\hat{C}^{\Lambda,\Lambda_0}(p)$ is $O(4)$ invariant, smooth in some sense, and such that $\Lambda^3\partial_\Lambda \hat{C}^{\Lambda,\Lambda_0}(p)$ and $\Lambda_0^2\Lambda^2\partial_\Lambda\partial_{\Lambda_0}\hat{C}^{\Lambda,\Lambda_0}(p)$ (together will all necessary derivatives w.r.t. p) are exponentially decreasing when $|p|/\Lambda\to\infty$. The main result of the theorem is that for any $\mathtt{N},\mathtt{L}$ and any multi–index $w\in\mathbb{N}_0^{4(\mathtt{N}-1)}$, there exist a polynomial $\mathcal{P}_\mathtt{L}$ of degree $\le \mathtt{L}$ and with non–negative coefficients, as well as a set of weighted trees $\mathcal{T}_{\mathtt{N},2\mathtt{L},w}$, such that (when e.g. $\mathtt{N}\ge 4$)

$$(4)\quad \left|\partial_p^w \hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{\mathtt{N}\ge 4,\mathtt{L}}\left(p_{[\mathtt{N}-1]}\right)\right| \le \mathcal{P}_\mathtt{L}\left(\log_+\left(\frac{|p_{[\mathtt{N}-1]}|_\mu}{\kappa}\right), \log_+\frac{\Lambda}{\mu}\right) \sum_{T\in\mathcal{T}_{\mathtt{N},2\mathtt{L},w}} \prod_{i\in\mathcal{I}(T)} |k_i|_\Lambda^{-\theta(i)}$$

for any $\Lambda_0>0$, $0<\Lambda\le\Lambda_0$ and $p_{[\mathtt{N}-1]}\in\mathbb{R}^{4(\mathtt{N}-1)}$. In (4), $\mu>0$ is the renormalization scale; $|p_{[\mathtt{N}-1]}| := \sup_e |p_e|$; $|p|_\Lambda := \sup(\Lambda,|p|)$; $\log_+ x := \log\sup(1,x)$. $\kappa := \sup(\Lambda, \inf(\eta(p_{[\mathtt{N}-1]}),\mu)) > 0$ is defined in terms of a "dynamical IR cutoff" $\eta(p_{[\mathtt{N}-1]}) := \inf_{\emptyset\ne\mathbb{S}\subseteq[\mathtt{N}-1]} |\sum_{e\in\mathbb{S}} p_e|$ (positive for non–exceptional momenta). $\mathcal{I}(T)$ is the set of internal lines of the weighted tree T; k_i is the momentum flowing through the internal line i, and $\theta(i)>0$ is the total weight associated to i.

The sets $\mathcal{T}_{\mathtt{N},\mathtt{R},w}$ ($\mathtt{R}\in\mathbb{N}_0$) satisfy two properties; *nestedness:* $\mathcal{T}_{\mathtt{N},\mathtt{R},w}\subseteq\mathcal{T}_{\mathtt{N},\mathtt{R}+1,w}$; *saturation:* $\mathcal{T}_{\mathtt{N},\mathtt{R},w} = \mathcal{T}_{\mathtt{N},3\mathtt{N}-2,w}$ for any $\mathtt{R}\ge 3\mathtt{N}-2$. The set $\mathcal{T}_{\mathtt{N},\mathtt{R},w=0}$ (corresponding to the absence of derivatives w.r.t. external momenta) is defined as the set of all $T=(\tau,\rho)$ in which τ is a tree and $\rho:\mathcal{I}(T)\to\{1,2\}$ is a line weight, such that: *a)* τ has $\mathtt{N}$ external lines and vertices of coordination number in $\{3,4\}$; *b)* the number of vertices with coordination 3 is $\le\mathtt{R}$; *c)* $\sum_{i\in\mathcal{I}(T)}\rho(i) = \mathtt{N}-4$; *d)* there is a bijection among the vertices of coordination number 3 and the internal lines with $\rho=1$. In the case $w=0$ one has $\theta(i)=\rho(i)$.

As an example, for any $\mathtt{L} > 0$ the set $\mathcal{T}_{\mathtt{N}=6,\mathtt{R}=2\mathtt{L},w=0}$ contains only the trees



and the trees derived from them by non–trivial permutations of the external momenta $p_{[0:5]}$. (Other trees with $\mathtt{N}$ external lines and vertices of coordination numbers $3, 4$ exist but do not satisfy to the defining conditions.) Correspondingly, in this case the bound (4) reads for any $\mathtt{L} > 0$

$$|\hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{6,\mathtt{L}}(p_{[5]})| \leq \big(|p_1+p_2+p_3|_\Lambda^{-2}+|p_1+p_2+p_3|_\Lambda^{-1}|p_4+p_5|_\Lambda^{-1}+|p_1+p_2|_\Lambda^{-1}|p_3+p_4|_\Lambda^{-1}+\text{perms.}\big)\,\mathcal{P}_{\mathtt{L}},$$

where $\mathcal{P}_{\mathtt{L}}$ has been introduced in (4).

The proof of the theorem is based on the recursive structure of the perturbative RG equations (3) (see e.g. [4]). The main difficulty is to wisely deal with spurious exceptional momenta, in order to keep the bound finite in the IR limit.

In the flow $\mathcal{F}^{\Lambda,\Lambda_0}_{\mathtt{N},\mathtt{L},w}$, see (3), the term quadratic in Schwinger functions acts as a junction of the weighted trees T', T'' in the bounds, respectively, of $\hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{\mathtt{N}',\mathtt{L}'}$, $\hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{\mathtt{N}'',\mathtt{L}''}$. Now, the junction of two weighted trees happens to be a weighted tree of the appropriate class and the inductive bound for $\hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{\mathtt{N},\mathtt{L}}$ is then reproduced.

The linear term in $\mathcal{F}^{\Lambda,\Lambda_0}_{\mathtt{N},\mathtt{L},w}$ is more problematic, because it contains a loop integration which tends to destroy the tree structure of the bounds. The exponential fall–off in ℓ/Λ of the covariance allows to prove ([5],[1]) bounds of the form

$$\int \mathrm{d}^4\ell\; |\partial_\Lambda \hat{C}^{\Lambda,\Lambda_0}(\ell)| \prod_{j=1}^{n} |\ell + k_j|_\Lambda^{-\theta_j} \leq c\,\Lambda \prod_{j=1}^{n} |k_j|_\Lambda^{-\theta_j}, \tag{5}$$

which, roughly speaking, amount to "cut the loop" and to set $\ell = 0$ by deleting two external lines for each tree. This property makes the linear part of the flow more "tree friendly". The elimination of the unwanted Λ factor in (5) (using the bound $\Lambda \leq |k_{j'}|_\Lambda$ for some j'), and the integration over Λ (to recover Schwinger functions from the flow) are taken into account by eliminating the factors $|k_{j'}|_\Lambda^{-1}, |k_{j''}|_\Lambda^{-1}$ for each tree in the original bound of $\hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{\mathtt{N}+2,\mathtt{L}-1}$, which amounts to consider a subtraction of two units in the original weights: this procedure can be consistently implemented as a mapping among our classes of weighted trees.

The logarithms in (4) originate from the Λ integration of the flow for marginal and irrelevant Schwinger functions, as well as from the integral interpolating marginal Schwinger functions from the renormalization point to a generic one.

References

[1] R. Guida, C. Kopper, in preparation.
[2] G. Keller, C. Kopper, Commun. Math. Phys. **161** (1994) 515-532.
[3] J. Polchinski, Nucl. Phys. **B231** (1984) 269-295.
[4] V. F. Müller, Rev. Math. Phys. **15** (2003) 491. [hep-th/0208211].
[5] C. Kopper, F. Meunier, Annales Henri Poincaré **3** (2002) 435-449. [hep-th/0110120].